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The lee-wave régime for a slender body in a rotating flow

By JOHN W. MILES†
Institute of Geophysics and Planetary Physics, University of California, La Jolla

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The uniform motion of a closed, axisymmetric body along the axis of an unbounded, rotating, inviscid, incompressible fluid is considered on Long's hypotheses that: the flow is steady; the flow is uniform far upstream of the body; the inertial waves excited by the body cannot propagate upstream. The appropriate similarity parameters are k, an inverse Rossby number based on the body length, and δ, the slenderness ratio of the body. It is conjectured that an upper bound to the parametric régime in which the solution implied by Long's hypotheses remains valid, say $k\delta \equiv \kappa < \kappa_c$, is determined by the first occurrence, with increasing κ, of a local reversal of the flow.

A general solution for the stream function is established in terms of an assumed distribution of dipoles along the axis of the body. The disturbance upstream of the body is found to be proportional to the product of κ^2 and the dipole moment (total dipole strength) and to fall off only as the inverse distance, as compared with the inverse cube of the distance for a potential flow. The corresponding wave drag is found to depend on the power spectrum of the dipole distribution in the axial wave-number interval $(0, k)$ and to be a monotonically decreasing function of the axial velocity for fixed angular velocity. Asymptotic solutions for prescribed bodies are established in the following limits: (i) $k \to 0$ with δ fixed; (ii) $\delta \to 0$ with k fixed; (iii) $k \to \infty$ with $k\delta$ fixed. Both the upstream disturbance and the wave drag in the limit (i) depend essentially on the dipole moment of the body with respect to a uniform, potential flow. The limit (ii) is analogous to conventional slender-body theory and yields a dipole density that is proportional to the cross-sectional area of the body. The limit (iii) leads to a singular integral equation that is solved to determine κ_c and the dipole moment for a slender body.

The results are applied to a sphere and a slender ellipsoid. The upstream axial velocity and the drag coefficient based on Stewartson's results for a sphere are found to differ significantly from Maxworthy's (1969) measurements, presumably in consequence of viscous separation effects. Maxworthy's measured values of upstream axial velocity are found to agree with the theoretical values for an equivalent ellipsoid, based on the sphere plus its upstream wake, for $\kappa \lesssim \kappa_c$.

† Also Department of Aerospace and Mechanical Engineering Sciences.

1. Introduction

We consider (see figure 1) the uniform motion of a closed, axisymmetric body along the axis of an unbounded, rotating, inviscid, incompressible fluid on the basis of Long's hypotheses: ($\mathscr{H}_1$) the flow is steady; ($\mathscr{H}_2$) the flow is uniform at a sufficient distance upstream of the body; ($\mathscr{H}_3$) the inertial waves excited by the body cannot propagate upstream.

This problem was first stated completely (in a mathematical sense) by Long (1953); earlier work, going back some fifty years to the original investigations of Taylor and Proudman, is reviewed by Squire (1956) and Greenspan (1968). The following treatment closely parallels, but is less detailed than, a recent study of

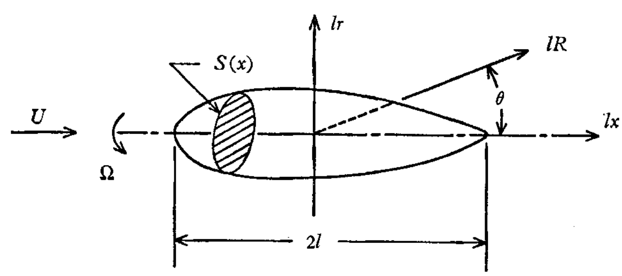

FIGURE 1. Geometrical configuration for body of revolution in rotating flow.

two-dimensional, stratified shear flow (Miles & Huppert 1969). It is aimed primarily at slender bodies, for which viscous effects may be of only secondary importance; however, we also reconsider and extend Stewartson's (1958) results for a sphere (for which viscous separation may be of dominant importance) in order to make comparisons with both Maxworthy's (1969) experimental results and the theoretical results for an ellipsoid.

The similarity parameters for flow around a body of prescribed shape, characteristic axial length l, and characteristic radius δl are the slenderness ratio δ, either of the inverse Rossby numbers

$$k = 2\Omega l/U \tag{1.1}$$

and

$$\kappa = k\delta, \tag{1.2}$$

and the drag coefficient

$$C_D(k, \delta) = (\tfrac{1}{2}\rho U^2 \pi \delta^2 l^2)^{-1} D, \tag{1.3}$$

where Ω is the angular velocity of the basic flow, U is the translational velocity of the body, ρ is the density of the fluid, and D is the wave drag on the body. We seek a description of the flow in that parametric range, say $0 < \kappa < \kappa_c(\delta)$, in which Long's model appears to be valid (see below) and focus especially on the lee-wave amplitude, the drag coefficient, and the upstream axial velocity as representative observables.

Validity of Long's hypotheses

A mathematically rigorous defence of $\mathscr{H}_1$ and $\mathscr{H}_2$ would appear to demand both the solution of a well-posed initial-value problem without the assumption of small disturbances (this assumption is not necessary in Long's model for steady flow) and a proof of the uniqueness and stability of the solution to that problem. No such solution is known, and it therefore is necessary to fall back on arguments that are, in some degree, either conjectural or controversial or both. The validity of $\mathscr{H}_3$ for waves of finite axial wavelength ($\alpha \neq 0$ in the subsequent development) follows from group-velocity arguments (Lighthill 1967), but these arguments do not exclude cylindrical waves ($\alpha = 0$), for which the group velocity may exceed the phase velocity (which must be equal to U for steady flow).

Long's hypotheses are implicitly accepted by Fraenkel (1956), who states that 'the upper limit of this range, where flow with no disturbances far upstream [breaks down]† is as yet unknown'; and, more tentatively, by Squire (1956), who states that 'waves are unlikely to arise ahead of the disturbance, if a solution without waves is possible'.

Stewartson, after considering the uniform translation of a sphere along the axis of a rotating fluid, concludes (1958) that 'a cylindrical component certainly occurs for [sufficiently large k], but may occur for all [k] > 0'. He has recently (1968*a*) given an alternative solution for the sphere on the hypothesis of separated flow and concludes that (personal communication) 'The results are not entirely satisfactory but it seems clear that the forward wake does occur for all [k] > 0'.

Trustrum (1964) considers the transient development of rotating or stratified flows on the hypothesis of small disturbances and concludes that 'the assumption of a uniform undisturbed upstream flow, which has been basic to most theories in both stratified flow and rotating fluids is probably not valid'. She notes, however, that 'the solution for a [plane] dipole with its axis along the direction of the uniform stream...has no terms independent of x and so its influence does not extend to upstream infinity'. We regard the implications of this last statement as decisive, at least for bodies of sufficiently small transverse dimensions, by virtue of the established fact that the steady motion associated with rotating flow past a closed, axisymmetric body can be attributed to an equivalent distribution of axial dipoles (see § 2 below) for sufficiently small values of κ.

Greenspan (1968) reviews the existing literature on the subject and concludes with the statement (p. 215):

'It seems likely that the hypothesis of no upstream disturbance is not strictly correct and that a body introduced into the stream will always affect the entry conditions at infinity to some degree. However, the assumption may be, and indeed probably is, an excellent *approximation* over a wide range of parameter settings because the amount of energy appearing upstream can be a small fraction of the total generated. Solutions derived on this basis can then be meaningful and significant.'

We take the position that Long's hypotheses are *strictly correct* for unseparated

† Square brackets within quotations indicate interpolations by the present writer.

flow in the sense that they yield an inner approximation, with a relative error of $O(E)$ as $E = 2\Omega\nu/U^2 \to 0$, that can be matched to an outer approximation that is determined by the Oseen approximation at distances of $O(l/kE)$ from the body.† It seems likely, however, that this inner approximation is unstable for sufficiently large κ, in which case steady, unseparated flow may be impossible.

Stability of inviscid, rotating flow

Long (1962) asserts that an inviscid, rotating flow that contains local reversals ($u < 0$) is necessarily unstable in consequence of the corresponding violation of Rayleigh's (1916) criterion that the square of the circulation be a monotonically increasing function of the cylindrical radius. It seems unlikely that Rayleigh's criterion, which refers essentially to static stability, is a sufficient condition for dynamic stability, but we accept Long's conjecture that it is a necessary condition. Accordingly, we limit our consideration of the lee-wave régime to $\kappa < \kappa_c$, where κ_c is the smallest value of κ for which $u = 0$ at one or more points in the flow outside of the body. We add that the solutions implied by Long's hypotheses typically contain closed streamlines for values of κ only slightly larger than κ_c. The effects of viscosity on the motion within these closed streamlines cannot be ignored, however large the Reynolds number (Batchelor 1956). Long's (1955) experiments on stratified flow suggest that the flow is locally turbulent in such regions, but that lee waves of decreasing amplitudes (relative to those for $\kappa < \kappa_c$) persist for $\kappa > \kappa_c$.

The boundary-value problem

We turn now to the formulation of the boundary-value problem implied by the hypotheses of the opening paragraph. Following Squire (1956), but using dimensionless variables and changing the sign of his perturbation stream function, we choose cylindrical polar co-ordinates $\{lx, lr, \phi\}$ in a reference frame in which the basic velocity is $\{U, 0, \Omega lr\}$ and the body is specified by $r = \delta\eta(x)$ and derive the perturbation velocity from a stream function $Ul\psi(x, r)$, such that the total velocity field is

$$\mathbf{v} \equiv \{u, v, w\} = \mathbf{V} + Ur^{-1}\{-\psi_r, \psi_x, -k\psi\}, \quad \mathbf{V} = \{U, 0, \Omega r\}. \tag{1.4}$$

Then ψ satisfies
$$\psi_{xx} + \psi_{rr} - r^{-1}\psi_r + k^2\psi = 0 \tag{1.5}$$

and the boundary conditions

$$\psi(x, r) = \tfrac{1}{2}r^2 \quad \text{on} \quad r = \delta\eta(x) \tag{1.6}$$

and
$$\psi = O(R^{-1}) \quad (x \to -\infty), \tag{1.7}$$

where
$$R = (x^2 + r^2)^{\frac{1}{2}} \tag{1.8}$$

is the spherical polar radius. The perturbation pressure is given by

$$p - p_0 = \tfrac{1}{2}\rho U^2\{2r^{-1}\psi_r - r^{-2}(\psi_x^2 + \psi_r^2 + k^2\psi)\}. \tag{1.9}$$

The uniqueness of the solution determined by (1.5)–(1.7) may be demonstrated

† I hope to explore this question further in a future paper.

by an appropriate modification of the proof for the corresponding, exterior boundary-value problem in classical wave theory [wherein (1.7) is replaced by Sommerfeld's radiation and finiteness conditions; see Courant & Hilbert 1962, pp. 312–20] if the fundamental solution in that problem is replaced by the dipole solution, ψ_1, given in § 2 below.

Outline of analysis

We begin our attack on the boundary-value problem by constructing, in § 2 below, a general solution of (1.5) and (1.7) in terms of an equivalent distribution, say $f(x)$, of dipoles along the axis of the body. The fundamental solution for this construction, say ψ_1, satisfies (1.5), exhibits a dipole singularity at a prescribed point on the axis, and satisfies (1.7) at infinity; it is due originally to Fraenkel (1956). We also determine a complete set of solutions, say $\{\psi_n\}$, from ψ_1 by axial differentiation (as in classical potential theory) and relate them to the corresponding set determined by Stewartson (1958). We then determine asymptotic approximations to the general solution in terms of $F(\alpha)$, the Fourier transform of $f(x)$ with respect to axial wave-number, and show that the upstream field depends essentially on the dipole moment, $F(0)$, the lee-wave field on $F(\alpha)$ in $\alpha = (0, k)$, and the wave drag on $|F(\alpha)|^2$ in $\alpha = (0, k)$.

In § 3, we apply the general results of § 2 to a sphere, for which the solution by separation of variables is given by Stewartson (1958). We use his results to determine the upstream axial velocity and the wave drag for $0 \leqslant k \leqslant \kappa_c$ and to determine $\kappa_c \doteqdot 2{\cdot}2$.

Stewartson's analysis could be extended to an ellipsoid, just as Huppert & Miles (1969) have extended Miles' (1968) analysis of stratified flow over a semi-circular cylinder to a semi-elliptical cylinder; however, such an extension would involve Lamé functions, which present computational difficulties. Analytical solutions for more general bodies appear to require further approximations.

We consider in § 4 the limit $k \to 0$ with δ fixed, which permits the flow in the neighbourhood of the body to be determined by potential theory and the asymptotic flow to be represented by a single dipole source, the strength of which is proportional to the dipole moment of the body with respect to a uniform potential flow. [This basic procedure is due to Rayleigh (1871, 1897) and is known as the *Rayleigh-scattering* approximation in diffraction theory.] We show that the limiting wave drag depends essentially only upon the square of this dipole moment.

We consider in § 5 the limit $\delta \to 0$ with k fixed, which permits the boundary condition (1.6) to be imposed on the axis, $r = 0$, rather than at $r = \delta\eta(x)$. This *slender-body* approximation, which has well-known antecedents in aerodynamics (Munk 1934; von Kármán 1936), implies that $f(x)$ is simply proportional to the cross-sectional area of the body and leads directly to a rather simple integral for the wave drag. The slender-body approximation may be extended by including higher-order terms in δ^2 and $\log \delta$ in the expansion of $\psi(x, \delta\eta)$ in (1.6), although it might be necessary to invoke the method of matched asymptotic expansions in consequence of the non-uniform validity of the expansion near the blunt ends of the body.

The slender-body approximation is subject to the implicit restriction $\kappa = k\delta \ll 1$ and is not uniformly valid as $k \to \infty$; accordingly, it cannot be used to obtain a reliable estimate of κ_c. We therefore consider the limit $k \to \infty$ with κ fixed in § 6. We find that the boundary condition (1.6) then yields a singular integral equation of Hilbert's type for $f(x)$, which we solve by invoking known results of function theory. This solution tends to the slender-body approximation as $\kappa \to 0$, which implies that the joint limit $k \to \infty$, $\delta \to 0$ is commutative.

We apply the results of §§ 5 and 6 to a slender ellipsoid in § 7 and calculate the wave drag in the slender-body approximation, the axial velocity far upstream of the body for $0 \leqslant k \leqslant \kappa_c$, and $\kappa_c \doteqdot 1{\cdot}94$. We find that the results for both the dipole moment (*upstream influence parameter*) and drag can be scaled in such a way as to yield curves that are close to the corresponding results for a sphere. We conjecture that these curves bound the correspondingly scaled results for other prolate ellipsoids ($0 < \delta < 1$). We expect the results for the ellipsoid to be qualititatively representative for other finite, slender bodies, with differences similar to those already established for several bodies in stratified flow (Miles & Huppert 1969).

Comparison with experiment

Maxworthy (1969) has recently measured the axial velocity upstream of, and the drag on, a sphere in a circular pipe of large radius for values of κ between 0·1 and 200 and values of $Re = 2Ua/\nu$ between 5 and 700. He reports that a finite slug of stagnant fluid appears in front of the sphere and that the length of this slug is a monotonically increasing function of κ for fixed κRe and of Re for fixed $\kappa > 2$ (the length of the slug appears to be independent of Re for $\kappa < 2$, but experimental scatter obscures this point).

We compare Maxworthy's measured values of the axial velocity upstream of the region of stagnant flow with the theoretical values deduced from Stewartson's (1958) solution in § 3 below and find substantial discrepancies. These discrepancies, which appear to reflect the dominant importance of viscous effects, suggest that an inviscid model is not likely to provide an adequate description of the flow past a body as bluff as a sphere in a rotating flow;† however, they throw little or no light on the validity of Long's hypotheses, $\mathscr{H}_1$–$\mathscr{H}_3$ above, for a flow, such as that past a slender body, in which viscous effects may be of only secondary importance.

We also (in § 7 below) compare Maxworthy's measured values of upstream axial velocity with the theoretical values, based on Long's model, for an ellipsoid having the same major and minor semi-axes as the sphere plus its upstream wake. The agreement is within the experimental scatter and suggests that Long's model may be adequate for the description of the flow past a slender body in the parametric range $\kappa \lesssim 2$, $\delta \lesssim 0{\cdot}5$.

† Long's (1953) original experiments, which provide qualitative confirmation of the adequacy of his model for the description of the lee-wave pattern, were carried out with a streamlined body having a spherical nose and a conical tail.

2. General solution

We pose a general solution of (1.5) and (1.7) in the form

$$\psi(x,r) = \int_{-\infty}^{\infty} f(\xi)\,\psi_1(x-\xi,r)\,d\xi, \tag{2.1}$$

where
$$f(x) = \psi(x,0+) \tag{2.2}$$

is an equivalent dipole density, and $\psi_1(x,r)$ is a fundamental solution of (1.5) that exhibits the dipole behaviour

$$\psi_1(x,r) \to \tfrac{1}{2}r^2(x^2+r^2)^{-\frac{3}{2}} = \tfrac{1}{2}R^{-1}\sin^2\theta \quad (kR\to 0) \tag{2.3a}$$

$$\to \delta(x) \qquad (r\to 0) \tag{2.3b}$$

at the origin and satisfies (1.7) at infinity; $\delta(x)$ is Dirac's delta function, and θ is the polar angle measured from the positive-x axis. We emphasize that $f(x)$ generally depends on both k and δ for a prescribed body and that it vanishes identically both outside of the body and over a finite interval of the interior axis in the neighbourhood of a blunt end.

The dipole solution ψ_1 has been determined by Fraenkel (1956, appendix III), who gives representations equivalent to the expansion of (3.2a) below and to

$$\psi_1(x,r) = \tfrac{1}{2}r\,\partial_r[-R^{-1}\cos kR + \int_0^k J_0\{(k^2-\alpha^2)^{\frac{1}{2}}r\}\sin\alpha x\,d\alpha] \tag{2.4a}$$

$$\sim kH(x)\sin kR\sin^2\theta \quad (kR\to\infty), \tag{2.4b}$$

where ∂_r implies partial differentiation with respect to r, and $H(x)$ is Heaviside's step function. Invoking the Fourier-integral representation (*Weyrich's formula*)

$$R^{-1}e^{ikR} = i\int_0^{\infty} H_0^{(1)}\{(k^2-\alpha^2)^{\frac{1}{2}}r\}\cos\alpha x\,d\alpha, \quad 0 \leqslant \arg(k^2-\alpha^2)^{\frac{1}{2}} < \pi, \tag{2.5}$$

we transform (2.4a) to

$$\psi_1(x,r) = -\tfrac{1}{2}r\,\partial_r\,\mathscr{R}\int_0^{\infty} iH_0^{(1)}\{(k^2-\alpha^2)^{\frac{1}{2}}r\}\,e^{i\alpha x}\,d\alpha, \tag{2.6}$$

where $\mathscr{R}$ implies *the real part of*, and the path of integration, say C, is indented under the branch point at $\alpha = k$ in an α-plane cut as shown in figure 2(a). The representation (2.6), which facilitates a Fourier-integral representation of the general solution (2.1), also may be obtained directly from the Fourier–Hankel transform solution of (1.5), (1.7) and (2.3b).

Substituting (2.6) into (2.1), we place the result in the form

$$\psi(x,r) = -\tfrac{1}{2}r\,\partial_r\,\mathscr{R}\int_0^{\infty} iF(\alpha)\,H_0^{(1)}\{(k^2-\alpha^2)^{\frac{1}{2}}r\}\,e^{i\alpha x}\,d\alpha, \tag{2.7}$$

where
$$F(\alpha) = \int_{-\infty}^{\infty} f(\xi)\,e^{-i\alpha\xi}\,d\xi \tag{2.8}$$

is the Fourier transform of $f(x)$.

We obtain an alternative representation of ψ, which separates the propagated from the non-propagated components of the total disturbance, by deforming the path C to $C_\pm$ for $\pm x > 0$, as shown in figures 2(*b*), (*c*), invoking the analytical continuation

$$H_0^{(1)}\{e^{i\pi}(k^2-\alpha^2)^{\frac{1}{2}}r\} = -H_0^{(2)}\{(k^2-\alpha^2)^{\frac{1}{2}}r\} \tag{2.9}$$

in the upper-half of the α-plane, introducing the change of variable $\alpha = i\eta\,\mathrm{sgn}\,x$ along the imaginary axis, and observing that $F(\pm i\eta)$ is real. The end result is

$$\psi(x,r) = -r\,\partial_r\Big[H(x)\,\mathscr{R}\int_0^k iF(\alpha)\,J_0\{(k^2-\alpha^2)^{\frac{1}{2}}r\}\,e^{i\alpha x}\,d\alpha + \tfrac{1}{2}\int_0^\infty F(i\eta\ \mathrm{sgn}\ x)\ J_0\{(k^2+\eta^2)^{\frac{1}{2}}r\}\ e^{-\eta|x|}\,d\eta\Big]. \tag{2.10}$$

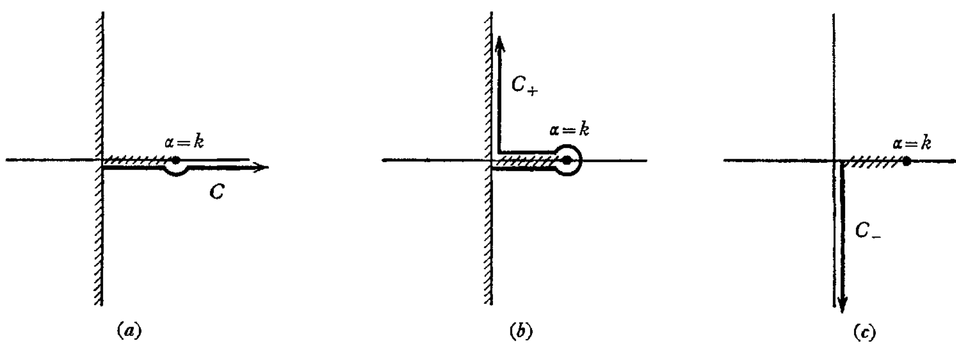

FIGURE 2. The contours C for (2.6) and (2.7), C_+ for (2.10) with $x > 0$, and C_- for (2.10) with $x < 0$.

The first integral comprises the inertial waves, which appear only in the lee of the body; the second integral comprises disturbances that decay exponentially in $|x|$ (but the integral superposition of these disturbances decays only like $1/|x|$; see below).

Multipole expansion

We obtain a third representation of $\psi(x,r)$ by expanding $\psi_1(x-\xi,r)$ about $\xi = 0$:

$$\psi(x,r) = \sum_{n=1}^{\infty} F_n\,\psi_n(x,r), \tag{2.11}$$

where
$$\psi_n(x,r) \equiv (-\partial_x)^{n-1}\{\psi_1(x,r)/(n-1)!\} \tag{2.12a}$$

$$\to \tfrac{1}{2}R^{-n}P_n'(\cos\theta)\sin^2\theta \quad (kR\to 0) \tag{2.12b}$$

$$\sim H(x)\sin^2\theta\cos^{n-1}\theta\,\mathscr{R}\{(-ik)^n\,e^{ikR}/(n-1)!\} \quad (kR\to\infty), \tag{2.12c}$$

P_n' is the derivative of a Legendre polynomial with respect to its argument, and

$$F_n \equiv \int_{-\infty}^{\infty} \xi^{n-1}f(\xi)\,d\xi \tag{2.13a}$$

$$= \{(i\,\partial_\alpha)^{n-1}F(\alpha)\}_{\alpha=0} \tag{2.13b}$$

is the nth moment of $f(x)$ and is, in general, a function of the parameters k and δ. We designate

$$F_1 = \int_{-\infty}^{\infty} f(x)\,dx = F(0) \tag{2.14}$$

as the *dipole moment*.†

The infinite set $\{\psi_n\}$, $n = 1, 2, \ldots$, is complete in $\theta = (0, \pi)$ for fixed R, and each of its members satisfies (1.5) and (1.7). It is linearly related to, but not identical with, the complete set $\{\hat{\psi}_n\}$ given by Stewartson (1958); see § 3 below.

Asymptotic representation

The dominant contributions to the asymptotic approximation to ψ as $kR \to \infty$ in the representation (2.7) come from the neighbourhoods of the points of stationary phase, if any, and the end-point $\alpha = 0$. Replacing $H_0^{(1)}$ by the dominant term in its asymptotic expansion, we find that the integrand has a point of stationary phase at $k = \alpha \cos\theta$ if and only if $x > 0$. Invoking the stationary-phase approximation to obtain the dominant contribution of this point and invoking Watson's lemma to obtain the dominant contribution from $\alpha = 0$, we obtain

$$\psi(x,r) \sim kH(x)\sin^2\theta\,\mathscr{R}\{F(k\cos\theta)\,e^{i(kR-\frac{1}{2}\pi)}\} - \tfrac{1}{2}kF_1\tan\theta J_1(kR\sin\theta) + O(R^{-1})$$
$$(kR \to \infty,\ |\tfrac{1}{2}\pi - \theta| > 0), \tag{2.15}$$

where F_1 is given by (2.14); this representation is not uniformly valid as $\theta \to \frac{1}{2}\pi$.

Substituting (2.15) into (1.4), we obtain the perturbation-velocity field

$$\begin{aligned}\mathbf{v} - \mathbf{V} \sim\ & k^2UH(x)R^{-1}\mathscr{R}[F(k\cos\theta)\,e^{ikR}\{-\sin\theta\cos\theta,\ \sin^2\theta,\ i\sin\theta\}] \\ & + \tfrac{1}{2}k^2F_1\,Ux^{-1}\{J_0(kr), 0, J_1(kr)\} \\ & + O(R^{-2}) \quad (kR \to \infty,\ |\tfrac{1}{2}\pi - \theta| > 0).\end{aligned} \tag{2.16}$$

The first term on the right-hand side of (2.16), which represents the inertial waves, is transverse to the spherical radius $\mathbf{R}$ and dominates the second term as $kR \to \infty$ in $0 < \theta < \frac{1}{2}\pi$. The magnitude of this transverse velocity, say $\tilde{v}$, is given by

$$\tilde{v} = k^2UR^{-1}|F(k\cos\theta)|\sin\theta. \tag{2.17}$$

The second term on the right-hand side of (2.15) makes only a negligible contribution to $\mathbf{v}$ in the limit $kR \to \infty$, $0 < \theta < \frac{1}{2}\pi$, but it dominates the asymptotic approximations to the radial velocity in $0 \leqslant \theta < \frac{1}{2}\pi$ and the total velocity in $\frac{1}{2}\pi < \theta \leqslant \pi$, both of which are $O(R^{-\frac{3}{2}})$ except near the axis ($\theta = 0$ or π). The perturbation velocity on the axis is given by

$$(u - U)/U \sim \mathscr{A}_1 x^{-1} \quad (r = 0,\ |x| \to \infty), \tag{2.18}$$

where
$$\mathscr{A}_1 = \tfrac{1}{2}k^2F_1 = \tfrac{1}{2}k^2F(0) \tag{2.19}$$

is, by definition, the *upstream-influence parameter*. We also remark that the axial perturbation velocity in $x < 0$ is oscillatory in r; in particular, the axial velocity in the direction of motion implied by (2.16) as $x \to \infty$ exceeds the velocity of the

† The moment defined by the right-hand side of (2.13*a*) is designated $\pi^{-1}F_{n-1}$, rather than F_n, in Miles & Huppert (1969).

body for $2{\cdot}4 < kr < 5{\cdot}5$ and has a maximum of $0{\cdot}4\mathscr{A}_1/|x|$ at $kr = 3{\cdot}8$. These last inferences are in qualitative agreement with Maxworthy's observations; see § 7 below.

Upstream influence

The result (2.18) is of special interest for the question of upstream influence, for it reveals that the axial perturbation velocity falls off much more slowly than in potential flow (for which $u - U \sim F_1 U/x^3$ on $r = 0$). We explore this question further by calculating the velocity at any point of the axis upstream of the body from (2.10), which reduces to

$$\psi(x,r) = -\tfrac{1}{2}r\,\partial_r \int_0^\infty F(-i\eta)\, J_0\{(k^2+\eta^2)^{\frac{1}{2}} r\}\, e^{\eta x}\, d\eta \quad (x < 0). \tag{2.20}$$

Invoking (1.4), (2.8) and (2.13), we obtain the representations

$$(u-U)/U = -\tfrac{1}{2}\int_0^\infty (k^2+\eta^2)\, F(-i\eta)\, e^{\eta x}\, d\eta \quad (x < 0,\ r = 0) \tag{2.21a}$$

$$= -\tfrac{1}{2}(k^2+\partial_x^2)\int_{-\infty}^{\infty} (\xi - x)^{-1} f(\xi)\, d\xi \tag{2.21b}$$

$$= \tfrac{1}{2}(k^2+\partial_x^2)\sum_{n=1}^{\infty} F_n x^{-n}. \tag{2.21c}$$

We observe that, by hypothesis, x lies outside of the ξ-range of integration, which lies within the body, and that the integral in (2.21*b*) is $\pi f_*(x)$, where $f_*(x)$ is the Hilbert transform of $f(x)$ (see § 6 below).

Letting $x \to -\infty$ in (2.21*c*), we recover (2.18) or, if we retain both components of the dipole term,

$$(u-U)/U \sim F_1(\tfrac{1}{2}k^2 x^{-1} + x^{-3}) \quad (x \to -\infty,\ r = 0), \tag{2.22}$$

which is uniformly valid as $k \to 0$, in which limit it tends to the axial velocity induced by a dipole in potential flow. We remark that the upstream flow ($x \to -\infty$) is dominated by the dipole component of ψ for all k, whereas the downstream flow is so dominated only for $1/x \ll k \ll 1$ (see § 4 below).

We note in passing that the counterpart of (2.21*b*) on the axis downstream of the body, as determined from (2.10), is

$$(u-U)/U = (k^2+\partial_x^2)\int_{-\infty}^{\infty} (x-\xi)^{-1}\{\tfrac{1}{2} - \cos k(x-\xi)\} f(\xi)\, d\xi \quad (x > 0, r = 0). \tag{2.23}$$

Letting $x \to \infty$ in (2.23), we again recover (2.18).

Wave drag

We calculate the wave drag by carrying out a momentum balance over a spherical surface of radius R and letting $R \to \infty$. Invoking the aforementioned, asymptotic properties of the velocity field, we find that the total transport of momentum across this surface vanishes asymptotically, in consequence of which the drag on the body is given by the axial thrust of the pressure on the surface,

$$D = -\lim_{R\to\infty} (2\pi l^2 R^2)\int_0^\pi p \sin\theta \cos\theta\, d\theta. \tag{2.24}$$

Substituting (1.9) into (2.24) and invoking (2.15), we find that the integrated contributions of p_0 and $r^{-1}\psi_r$ vanish identically and that the contributions of the quadratic terms to the integral over the upstream hemisphere vanish in the limit; the end result is

$$D = \pi\rho U^2 l^2 \lim_{R\to\infty} \int_0^{\frac{1}{2}\pi} (\psi_R^2 + k^2\psi^2) \cot\theta \, d\theta \tag{2.25a}$$

$$= \pi\rho U^2 l^2 k^4 \int_0^{\frac{1}{2}\pi} |F(k\cos\theta)|^2 \sin^3\theta \cos\theta \, d\theta. \tag{2.25b}$$

Invoking the change of variable $\alpha = k\cos\theta$ in (2.25*b*) and substituting the result into (1.3), we obtain

$$\delta^2 C_D = 2\int_0^k |F(\alpha)|^2 (k^2 - \alpha^2)\alpha \, d\alpha. \tag{2.26}$$

The function $|F(\alpha)|^2$ is the power spectrum of the dipole density and has the representations

$$|F(\alpha)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(\xi)\cos\{\alpha(x-\xi)\}\, d\xi\, dx \tag{2.27a}$$

$$= \left\{\sum_{m=0}^{\infty} (-)^m F_{2m+1}\alpha^{2m}/(2m)!\right\}^2 + \left\{\sum_{m=0}^{\infty} (-)^m F_{2m+2}\alpha^{2m+1}/(2m+1)!\right\}^2 \tag{2.27b}$$

by virtue of (2.8) and (2.13*b*), respectively. The drag depends only on that portion of the spectrum in $\alpha = (0, k)$ and therefore may be made arbitrarily small by a suitable tailoring of $f(x)$. We recall, however, that $f(x)$ depends on both k and δ except in the limit $\delta \to 0$, wherein it is proportional to the cross-sectional area of the body (see § 5 below); accordingly, it might be possible to design a body with zero wave drag (within the limitations of our idealized model) at one or more discrete values of k, but not for all k (cf. the Busemann biplane, which has zero wave drag at discrete values of the Mach number).

Substituting (2.27*a*) into (2.26) and integrating by parts with respect to ξ on the hypothesis that f vanishes at the end-points of the integration, we obtain

$$\delta^2 C_D = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f'(\xi)\,(\partial_x^2 + k^2)\left\{\frac{1-\cos k(x-\xi)}{x-\xi}\right\} d\xi\, dx. \tag{2.28}$$

Substituting (2.27*b*) into (2.26), we obtain the power series

$$\delta^2 C_D = \tfrac{1}{2}k^4\{F_1^2 + \tfrac{1}{3}(F_2^2 - F_1F_3)k^2 + \ldots\} \tag{2.29a}$$

$$= 2\mathscr{A}_1^2[1 + \tfrac{1}{3}\{(F_2/F_1)^2 - (F_3/F_1)\}k^2 + \ldots]. \tag{2.29b}$$

These alternative representations appear to be convenient for large and small k, respectively.

Substituting (2.26) and (1.1) into (1.3) and differentiating the result with respect to U, we find that D is a monotonically *decreasing* function of U for prescribed Ω and $F(\alpha)$. The implicit requirement that $F(\alpha)$ be independent of U is satisfied in the slender-body approximation of § 5; the analysis of § 6 suggests that the dependence of $F(\alpha)$ on k implies even larger values of $-\partial D/\partial U$ within the inertial-wave régime. The result (2.29*a*) implies that D vanishes like U^{-2} as $U \to \infty$.

3. Sphere

We consider, in terms of the general formulation of § 2, Stewartson's (1958) results for a sphere, for which $\delta \equiv 1$. We refer to equations in Stewartson's paper by the prefix S and replace W, l and ka in his notation by U, a and κ, respectively.

Stewartson poses the perturbation stream function in the form (S 4.2)

$$\psi = \tfrac{1}{2}\sum_{m=1}^{\infty} A_m \hat{\psi}_m(aR, \pi-\theta), \tag{3.1}$$

where $\hat{\psi}_m$ is given by (S 4.3) and satisfies (1.5) and (1.7). The function $\hat{\psi}_1$ is proportional to Fraenkel's dipole solution ψ_1; $\hat{\psi}_m$ is a linear combination of $\psi_1, \psi_2, \ldots, \psi_m$. Invoking the boundary condition (1.6), Stewartson obtains the infinite set of linear equations [(S 4.6) after correcting a typographical error]

$$\sum_{n=1}^{\infty} A_n\{\delta_{mn} J_{-\frac{1}{2}-m}(\kappa) + \alpha_{nm} J_{\frac{1}{2}+m}(\kappa)\} = \delta_{lm} \quad (m = 1, 2, \ldots), \tag{3.2}$$

where the numerical coefficients α_{nm} are determined by (S 3.11), (S 3.12), and (S 4.3) [P'_{2s} should be replaced by P'_{2s+2} in (S 3.12)].

Stewartson (1968*b*) gives numerical solutions of (3.2) for $\kappa = 1(1)6$ with four-figure accuracy for A_1, at least three-figure accuracy for A_2–A_4 (except for $\kappa = 1$ and 2, where both A_3 and A_4 are less than 0·01), and lesser accuracy for A_5–A_9. [Stewartson's (1958) original results are less accurate in consequence of a spurious singularity at $\kappa = 5{\cdot}76$, which appears to be near a zero (possibly complex) of the truncated determinant; cf. Miles (1968), where this difficulty is discussed in more detail.] We obtain analytical approximations by truncation. The first such approximation is

$$A_1^{(1)} = (\tfrac{1}{2}\pi/k)^{\frac{1}{2}}\{y_1(k)\}^{-1}, \tag{3.3}$$

where j_n (below) and y_n are spherical Bessel functions. The second approximation, obtained by retaining the first two terms in each of the first three equations ($m = 1, n = 1, 2$; $m = 2, n = 1, 2$; $m = 3, n = 2, 3$), is

$$\left.\begin{aligned} A_1^{(2)} &= 1 + (45 j_1 j_2/64 y_1 y_2)^{-1} A_1^{(1)}, \\ A_2^{(2)} &= -(5j_2/8y_2)A_1^{(2)}, \quad A_3^{(2)} = -(7j_3/16y_3)A_2^{(2)}, \end{aligned}\right\} \tag{3.4}$$

where, here and subsequently, the argument of each of the j_n and y_n is k. We find that the first and second approximations of (3.3) and (3.4) are within 1 % of Stewartson's results for $k = 1$ and $k = 2$, respectively. It is clear from the form of (3.2) that the accuracy of a given truncation decreases rapidly with increasing κ and that the first approximation breaks down completely at the smallest zero of $y_1(k)$, $k = 2{\cdot}80$ (the truncated determinant of second order does not appear to have a real zero in $k \leqslant 4{\cdot}3$; $k = 4{\cdot}3$ is the smallest real zero of the truncated determinant of third order).

We relate the A_n in the expansion of (3.1) to the F_n in the expansion of (2.11) by comparing the asymptotic representation implied by (3.1) and Stewartson's asymptotic representations of the $\hat{\psi}_m$ to that implied by (2.11) and (2.12*c*).

Equating the coefficients of $\cos kR$ and $\sin kR$ in these representations and writing $\mu = \cos\theta$, we obtain (for $0 < \mu \leqslant 1$)

$$-(\tfrac{1}{2}\pi k^3)^{-\frac{1}{2}} \sum_{m=0}^{\infty} (-)^m A_{2m+1} P'_{2m+1}(\mu) = \sum_{n=0}^{\infty} (-)^n \{k^{2n}/(2n)!\} F_{2n+1} \mu^{2n} \tag{3.5a}$$

and

$$-(\tfrac{1}{2}\pi k^3)^{-\frac{1}{2}} \sum_{m=1}^{\infty} (-)^m A_{2m} P'_{2m}(\mu) = \sum_{n=1}^{\infty} (-)^n \{k^{2n-1}/(2n-1)!\} F_{2n} \mu^{2n-1}. \tag{3.5b}$$

We reduce (3.5a, b) to an infinite set of linear equations with constant coefficients by invoking the orthogonality of the independent, complete sets $(1-\mu^2)^{\frac{1}{2}} P'_{2m}(\mu)$ and $(1-\mu^2)^{\frac{1}{2}} P'_{2m+1}(\mu)$ in $\mu = (0, 1)$. Retaining only the first three of each of the A_n and the F_n, we obtain

$$\{F_1, F_2, F_3\} = -(\tfrac{1}{2}\pi k^3)^{-\frac{1}{2}} \{A_1 + \tfrac{3}{2}A_3, 3k^{-1}A_2, 15k^{-2}A_3\} \tag{3.6a}$$

$$= (-k^2 y_1)^{-1} [1 + (45 j_1 j_2 / 64 y_1 y_2)]^{-1}$$
$$\times \{1 + (105 j_2 j_3 / 256 y_2 y_3), (-15 j_2 / 8 k y_2), (525 j_2 j_3 / 128 k^2 y_2 y_3)\}, \tag{3.6b}$$

where (3.6b) follows from (3.6a) through (3.4). The moments given by (3.6b) are plotted in figure 3. The corresponding upstream-influence parameter of (2.19) is plotted in figure 4.

Retaining the first three modes in (3.1), we find that the restriction $u > 0$ is first violated near $x = 0+$ and $kr = 2\pi$ for $k = \kappa_c = 2{\cdot}2$; a calculation based on the first mode alone yields $\kappa_c \doteqdot 2{\cdot}1$. The results for the analogous problem of a semi-circular obstacle in a stratified flow (Miles 1968) suggest that more accurate calculations would move the critical point somewhat downstream of the equator without significantly altering κ_c.

We approximate the velocity on the upstream axis by retaining only the dipole ($n = 1$) and quadrupole ($n = 2$) terms in the expansion of (2.21c) and rewriting the result in the form

$$u_1 \equiv (U-u)/U = \mathscr{A}_1 R^{-1} \{1 - (F_2/F_1) R^{-1}$$
$$+ (2/k^2) R^{-2} - (6/k^2)(F_2/F_1) R^{-3}\} \quad (R \geqslant 1,\ \theta = \pi), \tag{3.7}$$

which represents the axial velocity, in the direction of motion, forward of a sphere moving with unit velocity.† Substituting the numerical values of F_1 and F_2 given by (3.6b) into (3.7), we obtain the results plotted in figure 5; they are within 1 % of the known value at the boundary, $u_1 = 1$ at $R = 1$, for $k \leqslant 2{\cdot}2$. We also find that the dipole approximation of (2.18),

$$u_1 \sim \mathscr{A}_1 R^{-1} \quad (\theta = \pi), \tag{3.8}$$

differs from (3.7) by less than 5 % for $R \geqslant 3$ and $k \leqslant 2{\cdot}2$.

† Stewartson's representation of the $\hat{\psi}_m$ is not well suited to the calculation of the upstream field ($kR \to \infty$, $\theta > \frac{1}{2}\pi$) in consequence of a rate of convergence that decreases with increasing kR (for $\frac{1}{2}\pi < \theta \leqslant \pi$) in a way that is reminiscent of the separation-of-variables solution for scattering by a sphere when its radius is large compared with the wavelength of the scattered wave. This difficulty could be overcome by a Watson transformation, which would yield results equivalent to those determined by the asymptotic development of the representation (2.20)—in particular, (2.21c).

Substituting the numerical values given by (3.6*b*) into (2.28*a*) we obtain the drag coefficient plotted in figure 6; it is within 0·5 % of Stewartson's result for $k = 2$. We also find that contribution of the octupole ($n = 3$) to C_D is less than 1 % for $k \leqslant 2$ and that the dipole approximation,

$$C_D \doteqdot 2\mathscr{A}_1^2, \tag{3.9}$$

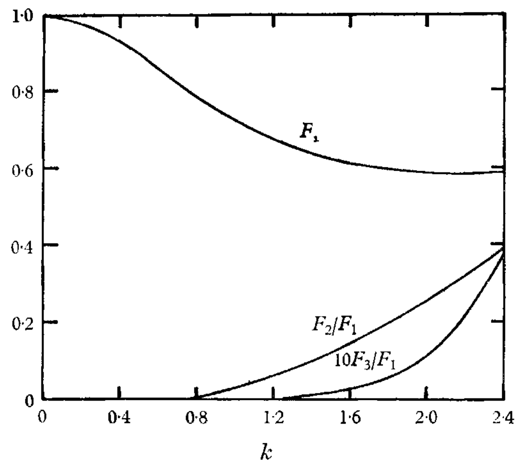

FIGURE 3. The moment F_1 and the moment ratios F_2/F_1 and $10F_3/F_1$ for a sphere, as given by (3.6*b*).

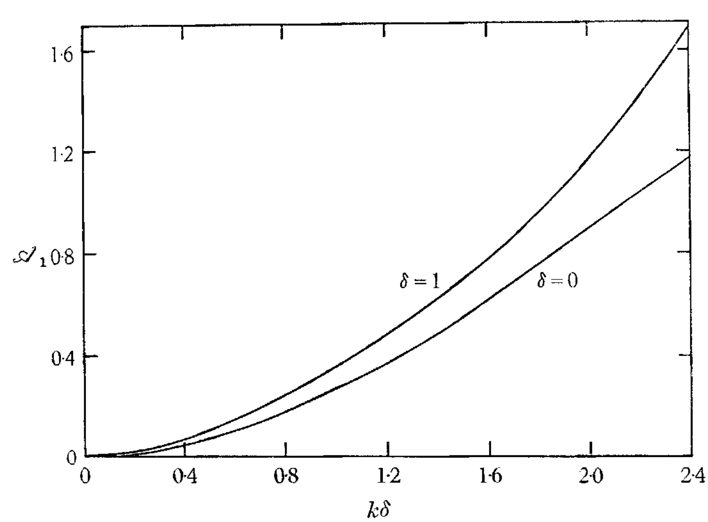

FIGURE 4. The upstream-influence parameters for a prolate ellipsoid, as determined from (2.19) and (3.6*b*) for $\delta = 1$ (sphere) and (7.14) for $\delta \to 0$.

differs from the result of figure 4 by less than 1 % for $k \leqslant 1·5$ and less than 10 % for $k \leqslant 2$.

Maxworthy's (1969) measured values of u_1 for $k = 1·74$ and 2·16 in the range $3 < R < 10$ are roughly double those given by (3.7). His measured values of C_D are independent of $Re \equiv 2Ua/\nu$ for large Re and $k > 5$. They also are independent of Re for smaller values of k after extrapolation from the actual experimental values to larger values of Re; for example, extrapolation yields $C_D = 1{\cdot}8 \pm 0{\cdot}3$ at $k = 2$, which is substantially smaller than Stewartson's (1958) theoretical value of 2·8. These discrepancies between theory and experiment appear to be associated with the presence of viscous wakes both fore and aft of the sphere. They render it fairly certain that an inviscid model for a body as bluff as a sphere in a rotating flow is quite inadequate within the parametric range of Maxworthy's experiments and suggest that it is unlikely to be adequate for any parametric range.

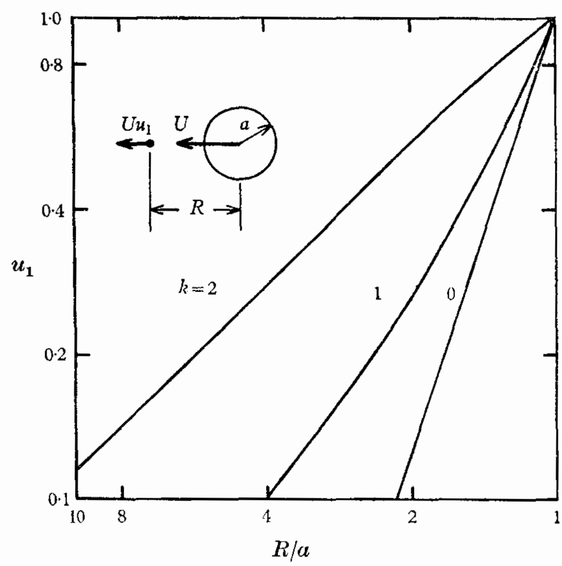

FIGURE 5. The axial velocity upstream of a sphere, as given by (3.7).

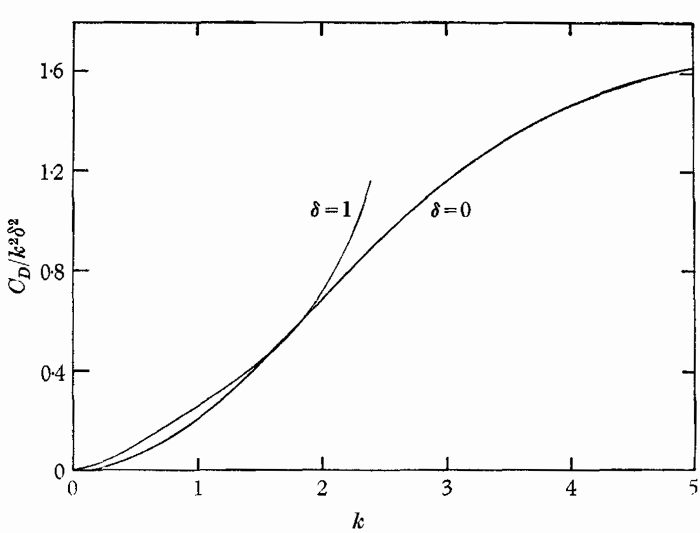

FIGURE 6. The drag coefficients for a prolate ellipsoid as given by (2.29) and (3.6) for $\delta = 1$ (sphere) and by (7.5a) for $\delta \to 0$ (slender-body approximation).

4. Rayleigh-scattering approximation ($k \to 0$)

Letting $k \to 0$ in (2.12*b*) and (2.12*c*) and substituting the results into (2.11), we obtain the inner and outer asymptotic approximations

$$\psi(x,r) \sim \tfrac{1}{2}\sin^2\theta \sum_{n=1}^{\infty} F_n R^{-n} P_n'(\cos\theta)\{1+O(k^2R^2)\} \quad (kR \to 0) \tag{4.1a}$$

$$\sim kF_1 H(x)\sin kR \sin^2\theta\{1+O(k)\} \quad (kR \to \infty). \tag{4.1b}$$

The corresponding approximation to the drag coefficient is given by (2.29*a*), which reduces to

$$\delta^2 C_D = \tfrac{1}{2}k^4 F_1^2\{1+O(k^2)\}, \tag{4.2}$$

wherein F_1 is evaluated for $k = 0$ (potential flow). It would be consistent with the inner approximation, (4.1*a*), to include the quadrupole term ($n = 2$) in the outer approximation, (4.1*b*), but this term makes no contribution to the drag within the overall error factor of $1+O(k^2)$.

The flow in the neighbourhood of the body is potential within $1+O(k^2)$ and may be determined by the methods developed for airships (Munk 1934; von Kármán 1927; Taylor 1928). The results so determined generally will not be in the form (4.1*a*), which may not converge in $R < 1$, but may be placed in that form by analytic continuation.

We infer from (4.1*b*) and (4.2) that the wave field downstream of the body and the resulting drag are determined essentially by the dipole moment of the body with respect to a uniform potential flow. The properties of this dipole moment are discussed by Lamb (1932, § 121*a*), Taylor (1928), and Pölya & Szegö (1951). Perhaps the most important of these properties are that

$$F_1 = (V+W)/2\pi l^3 \quad (k = 0), \tag{4.3}$$

where V and ρW are the volume and the axial component of the virtual mass of the body, and that l^3F_1 is a monotonically increasing set function of the boundary. We may use the latter property to bound l^3F_1 from above and below with the aid of the known results for ellipsoids of revolution. We also remark that F_1 is invariant under a reversal of the flow, by virtue of which the limiting drag (as $k \to 0$) is similarly invariant.

Modifying a suggestion by Munk (1934), we suggest that the dipole moment of any smooth, prolate body of revolution of length $2l$ and maximum diameter $2\delta l$ may be approximated by the result for an ellipsoid (Lamb 1932, § 114) according to

$$2\pi l^3 F_1/V = (1-\delta^2)[1-\delta^2(1-\delta^2)^{-\frac{3}{2}}\log\{(1+(1-\delta^2)^{\frac{1}{2}})/\delta\}]^{-1} \quad (0 < \delta < 1) \tag{4.4a}$$

$$= 1+\delta^2\log(1/\delta)+O(\delta^2) \quad (\delta \to 0). \tag{4.4b}$$

$$\doteqdot 0{\cdot}92+0{\cdot}58\delta \quad (0{\cdot}4 \leqslant \delta \leqslant 1). \tag{4.4c}$$

The approximation (4.4*b*) is within 1 % of the exact result (4.4*a*) for $\delta \leqslant 0{\cdot}4$ [including the term of $O(\delta^2)$ in (4.4*b*) actually degrades the approximation for

$\delta > 0{\cdot}1$]. The empirical approximation (4.4*c*) is within 1 % of the exact result (4.4*a*) for $0{\cdot}4 \leqslant \delta \leqslant 1$.

5. Slender-body approximation ($\delta \to 0$)

Invoking the boundary condition (1.6) in (2.1), we obtain the integral equation

$$\int_{-\infty}^{\infty} f(\xi)\,\psi_1\{x-\xi,\,\delta\eta(x)\}\,d\xi = \tfrac{1}{2}\delta^2\eta^2(x) \equiv S(x)/2\pi l^2, \tag{5.1}$$

where $S(x)$ is the cross-sectional area of the body. Letting $\delta \to 0$ and invoking the limiting approximation (2.3*b*) for ψ_1, we obtain

$$f(x) = (2\pi l^2)^{-1} S(x)\,\{1 + O(\delta^2 \log \delta)\} \quad (\delta \to 0,\ k \text{ fixed}). \tag{5.2}$$

The approximation (5.2) is not uniformly valid either for $k \to \infty$ or in the neighbourhood of a blunt end, where $|\eta'(x)| \to \infty$. A modified slender-body approximation that is valid as $k \to \infty$ with $k\delta$ fixed is developed in § 6. An approximation that is uniformly valid in the neighbourhoods $|x \mp 1| \ll 1/k$ for locally paraboloidal ends may be inferred from the work of Moran (1963) and Handelsman & Keller (1967).

The multipole moments of (2.13) are of limited interest for slender bodies, and we notice only that the substitution of (5.2) into (2.14) yields

$$F_1 = (V/2\pi l^3)\,\{1 + O(\delta^2 \log \delta)\} \quad (\delta \to 0), \tag{5.3}$$

in agreement with the result obtained by letting $\delta \to 0$ in (4.3); however, (5.3) requires only $k\delta \ll 1$, rather than $k \ll 1$, for its validity.

Substituting (5.2) into (2.28) and invoking (1.3), we obtain the slender-body approximation to the wave drag in the form

$$D = \frac{\rho U^2}{4\pi l^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x)\,S'(\xi)\,(\partial_x^2 + k^2) \times \left\{\frac{1 - \cos k(x-\xi)}{x-\xi}\right\} d\xi\, dx\,\{1 + O(\delta^2 \log \delta)\}. \tag{5.4}$$

The error term is for a slender body that is no more blunt than an ellipsoid at both bow and stern.

Alternative forms for the drag may be obtained from (5.4) by integration by parts. The most convenient form for the actual calculation, as in § 7 below, may be that of (2.26). We infer from (5.4) that the drag of a slender body ($\delta \to 0$) is invariant under a reversal of the flow independently of axial symmetry.

Letting $k \to \infty$ in (5.4), invoking the Riemann–Lebesgue lemma, integrating by parts, and substituting k from (1.1), we obtain

$$D \sim (\rho\Omega^2/\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S'(x)\,S'(\xi) \log |x-\xi|^{-1}\, d\xi\, dx \quad (\delta \to 0,\ k \to \infty), \tag{5.5}$$

which is analogous to the well-known results of Prandtl for the vortex drag on a lifting line and of von Kármán for the supersonic wave drag on a body of revolution [but note that this last result contains $\tfrac{1}{2}US''(x)$ in place of $\Omega S'(x)$]. We infer from this analogy that minimizing (5.5) under the constraints of prescribed length and volume, say $2l$ and V, yields the elliptic cross-sectional distribution

$$S(x) = S_0(1-x^2)^{\frac{1}{2}} H(1-|x|) \tag{5.6}$$

and
$$D = \tfrac{1}{2}\pi\rho\Omega^2 S_0^2 = (2/\pi)(\rho\Omega^2 V^2/l^2) \quad (\delta \to 0,\ k \to \infty). \tag{5.7}$$

[The body described by (5.6) is a *Kármán ogive* reflected in $x = 0$ and is slightly more blunt than an ellipsoid. Von Kármán (1936) gave the result for the half-body of prescribed length and calibre with minimum, supersonic wave drag.]

The limiting result (5.7) is of little direct value for even very slender bodies (see §§ 6 and 7 below), but it does provide a convenient basis of comparison for bodies of prescribed volume. The minimization of D for prescribed frontal cross-section, S_0, is more difficult, but we remark that the infinitely long, spindle-like body prescribed by

$$S(x) = S_0(1+x^2)^{-1} \tag{5.8}$$

yields
$$D \sim \tfrac{1}{4}\pi\rho\Omega^2 S_0^2 \quad (\delta \to 0,\ k \to \infty), \tag{5.9}$$

which is only half that given by (5.7) for equal S_0.

6. Low-speed, slender-body approximation ($k \to \infty$, $\delta \to 0$)

We now construct a singular integral equation for $f(x)$ in the limit $k \to \infty$ with $\kappa = k\delta$ fixed and reduce its solution to the solution of the Dirichlet problem for a half-plane.

We begin with the following definitions: (i) $\mathscr{C}$ is a class of functions of the real variable x that are continuous and belong to $L^2(-\infty, \infty)$. (ii) $\mathbf{C}$ is a class of functions of the complex variable $\mathbf{x} = x + ix_i$ that are holomorphic in the half-plane $x_i > 0$ and $O(1/|\mathbf{x}|)$ as $|\mathbf{x}| \to \infty$ in $x_i > 0$. (iii) The Hilbert transform (Titchmarsh 1948, chapter 5) of $f(x)$ is given by

$$f_*(x) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi - x} \quad (f \text{ in } \mathscr{C}), \tag{6.1}$$

where the Cauchy principal value of the integral is implied by the crossed integral sign; $f_*(x)$ is in $\mathscr{C}$ if $f(x)$ is in $\mathscr{C}$. (iv) The Cauchy integral of $f(x)$, given by

$$\mathbf{f}(\mathbf{x}) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi - \mathbf{x}} \quad (x_i > 0), \tag{6.2}$$

is in $\mathbf{C}$ and reduces to $\quad \mathbf{f}(\mathbf{x}) = f(x) - if_*(x) \quad (x_i = 0+) \qquad (6.3)$

on the real axis. Accordingly, $\mathbf{f}(\mathbf{x})$ is a solution to the Dirichlet problem for prescribed $f(x)$ in $\mathscr{C}$ [the most general solution is $\mathbf{f} + iC$, where C is a real constant that vanishes identically in the present context; see Titchmarsh (1948, chapter 5) and Muskhelishvili (1953, chapter 2) for more general discussions].

We determine the asymptotic approximation to ψ_1 by replacing x by $\mathbf{x}$ in (2.6), introducing the change of variable $\alpha = k\beta$, integrating by parts along a path indented under $\beta = 1$ (the integral is absolutely convergent for $x_i > 0$) and letting $k \to \infty$ while holding kr fixed:

$$\psi_1(\mathbf{x}, r) = -\tfrac{1}{2}kr\,\partial_r \mathscr{R}\int_0^\infty iH_0^{(1)}\{kr(1-\beta^2)^{\frac{1}{2}}\}\,e^{ik\mathbf{x}\beta}\,d\beta \tag{6.4a}$$

$$\sim -\tfrac{1}{2}kr\mathscr{R}\{H_1^{(1)}(kr)\,\mathbf{x}^{-1}\} \quad (x_i > 0,\ k\to\infty). \tag{6.4b}$$

The error factor for (6.4*b*) is $1+O(1/k)$, uniformly with respect to kr. Substituting (6.4*b*) into (2.1) and invoking (6.2), we obtain

$$\psi \sim \mathscr{R}\{\tfrac{1}{2}i\pi kr H_1^{(1)}(kr)\,\mathbf{f}(\mathbf{x})\} \quad (x_i = 0+) \tag{6.5a}$$

$$= \tfrac{1}{2}\pi kr\{-f(x)\,Y_1(kr) + f_*(x)\,J_1(kr)\}. \tag{6.5b}$$

Setting $r = \delta\eta$ in (6.5*a*), invoking (1.6), and dividing the result by two, we obtain the singular integral equation

$$\mathscr{R}\{\Omega(x)\,e^{i\omega(x)}\,\mathbf{f}(\mathbf{x})\} = \tfrac{1}{2}\delta^2\eta^2(x) \quad (x_i = 0+), \tag{6.6}$$

where

$$\Omega(x) = \tfrac{1}{2}\pi\kappa\eta\{J_1^2(\kappa\eta) + Y_1^2(\kappa\eta)\}^{\frac{1}{2}} \tag{6.7a†}$$

$$\to 1 - \tfrac{1}{2}(\kappa\eta)^2\{\log(\tfrac{1}{2}\kappa\eta) + \gamma - \tfrac{1}{2}\}\,(\kappa\eta\to 0) \tag{6.7b}$$

$$\sim (\tfrac{1}{2}\pi\kappa\eta)^{\frac{1}{2}} \quad (\kappa\eta\to\infty) \tag{6.7c}$$

and

$$\omega(x) = \tan^{-1}\{-J_1(\kappa\eta)/Y_1(\kappa\eta)\} \tag{6.8a}$$

$$\to \tfrac{1}{4}\pi\kappa^2\eta^2 \quad (\kappa\eta\to 0) \tag{6.8b}$$

$$\sim \kappa\eta - \tfrac{1}{4}\pi \quad (\kappa\eta\to\infty). \tag{6.8c}$$

We also require

$$\omega'(x) = \tfrac{1}{2}\pi\kappa^2\eta(x)\,\eta'(x)/\Omega^2(x), \tag{6.9}$$

which follows from (6.8*a*) by virtue of the Wronskian relation between J_1 and Y_1.

Muskhelishvili (1953, § 47) gives a general solution to a singular integral equation that is equivalent to (6.6), but we find it economical to proceed independently. Multiplying (6.6) through by $\exp(\omega_*)/\Omega$ and introducing

$$g(x) = \tfrac{1}{2}\delta^2\eta^2(x)\,e^{\omega_*(x)}/\Omega(x), \tag{6.10}$$

we obtain

$$\mathscr{R}\{e^{i\omega(\mathbf{x})}\,\mathbf{f}(\mathbf{x})\} = g(\mathbf{x}) \quad (x_i = 0+). \tag{6.11}$$

We may show that $\omega(x)$ and $g(x)$ are in $\mathscr{C}$, and hence that $\boldsymbol{\omega}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are in $\mathbf{C}$; accordingly, we may continue (6.11) into $x_i > 0$ to obtain

$$\mathbf{f}(\mathbf{x}) = e^{-i\omega(\mathbf{x})}\,\mathbf{g}(\mathbf{x}) \quad (x_i > 0). \tag{6.12}$$

Setting $x_i = 0+$ in (6.12) and taking the real and imaginary parts of the result, we obtain

$$f(x) = \tfrac{1}{2}\delta^2(\lambda\cos\omega + \mu\sin\omega) \tag{6.13}$$

and

$$f_*(x) = \tfrac{1}{2}\delta^2(\lambda\sin\omega - \mu\cos\omega), \tag{6.14}$$

where

$$\lambda(x) = \eta^2(x)/\Omega(x) \tag{6.15}$$

and

$$\mu(x) = -e^{-\omega_*(x)}\,(e^{\omega_*}\eta^2/\Omega)_*. \tag{6.16}$$

† The function $\Omega(x)$ is not related to the angular velocity Ω, which appears in the analysis only implicitly through k and κ.

We remark that both λ and ω, and hence also f, vanish like η^2 at the end points of the body, but that (6.13) and (6.14) do not yield uniformly valid approximations to the velocity in the neighbourhood of blunt ends.

Turning to the questions of stability and upstream influence, we substitute (6.5 *b*) into (1.4) to obtain

$$u/U = 1 - \tfrac{1}{2}\pi k^2\{-f(x)Y_0(kr) + f_*(x)J_0(kr)\} \quad (k \to \infty). \tag{6.17}$$

Setting $\eta(x) = 0$ and $r = 0$ in (6.13), (6.14), and (6.17), we obtain [cf. (2.18), (2.21 *b*), and (2.23)]

$$u/U = 1 + \tfrac{1}{4}\pi\kappa^2\mu(x) \quad (r = 0) \tag{6.18}$$

which implies

$$\mathscr{A}_1 = \tfrac{1}{4}\kappa^2\int_{-\infty}^{\infty}\{e^{\omega_*(\xi)}\eta^2(\xi)/\Omega(\xi)\}\,d\xi. \tag{6.19}$$

Letting kr and $\kappa\eta$ tend to zero in (6.17), we find that $u > 0$ in the neighbourhood of the body ($\delta\eta < r \ll 1/k$) for sufficiently small κ, and that u first vanishes with increasing κ at some point well removed from the body. The first critical point for a symmetric body of finite length, for which $\eta \equiv 0$ in $|x| \geqslant 1$ and $\eta(-x) = \eta(x)$, appears to arise at $x = 1+$ and $kr = 3{\cdot}83$ (where $f = 0$ and J_0 has its first minimum), in which case κ_c is given by ($x = 1+$ implies $\omega = 0$ and $f_* = -\tfrac{1}{2}\delta^2\mu$)

$$0{\cdot}101\pi\kappa^2\mu(1) = 1 \quad (\kappa = \kappa_c), \tag{6.20}$$

which is an implicit equation for κ_c.

The analogous example of a semi-elliptical obstacle in a stratified shear flow (Huppert & Miles 1969) suggests that the critical point at which $u > 0$ is first violated always lies downstream of the equator for a symmetric body and moves to the plane of the stern ($x = 1+$) as $\delta \to 0$. The examples considered by Miles & Huppert (1969) suggest that the first critical point for a finite, *asymmetric*, slender body may arise in $|x| < 1$, but even then (6.20) would give an upper bound to κ_c.

We obtain the asymptotic approximation to the drag by letting $k \to \infty$ in (2.28) and invoking the Riemann–Lebesgue lemma and (6.1):

$$C_D = (2\pi k^2/\delta^2)\int_{-\infty}^{\infty} f'(x)f_*(x)\,dx. \tag{6.21}$$

Substituting (6.13) and (6.14) into (6.21), we obtain

$$C_D = \tfrac{1}{4}\pi\kappa^2\int_{-1}^{1}[2\lambda(x)\mu'(x) - \omega'(x)\{\lambda^2(x) + \mu^2(x)\}]\,dx. \tag{6.22}$$

7. Slender ellipsoid

We illustrate the results of §§ 6 and 7 by considering a slender, prolate ellipsoid of length $2l$ and transverse diameter $2\delta l$, for which

$$\eta = (1 - x^2)^{\frac{1}{2}}H(1 - |x|). \tag{7.1}$$

Invoking (5.2) and (2.8), we obtain the slender-body approximation

$$F(\alpha) = 2\delta^2\alpha^{-3}(\sin\alpha - \alpha\cos\alpha) \tag{7.2 a}$$

$$= \tfrac{2}{3}\delta^2(1 - \tfrac{1}{10}\alpha^2 + \tfrac{1}{140}\alpha^4 + \ldots) \tag{7.2 b}$$

within $1+O(\delta^2 \log \delta)$. The first two terms in (7.2*b*) give $F(\alpha)$ within 10% for $\alpha \leqslant 2$. The first zero of $F(\alpha)$ is $\alpha = 4{\cdot}49$. Substituting $F(0)$ into (2.19), we obtain

$$\mathscr{A}_1 = \tfrac{1}{3}\kappa^2\{1+O(\delta^2 \log \delta)\}. \tag{7.3}$$

Substituting (7.2*a*, *b*) into (2.15), we obtain the following approximations to the lee-wave field:

$$\psi \sim k \sin kR \sin^2\theta F(k\cos\theta) \quad (kR \to \infty) \tag{7.4 a}$$

$$= \tfrac{2}{3}k\delta^2 \sin kR \sin^2\theta\{1-\tfrac{1}{10}k^2\cos^2\theta+\ldots\}\{1+O(\delta^2\log\delta)\}. \tag{7.4 b}$$

Substituting (7.2) into (2.26) and reducing the integral, we obtain

$$C_D = 2\delta^2\left\{\frac{\sin^2 k}{k^2}-\frac{\sin 2k}{k}+1+k^2-2\int_0^{2k}\left(\frac{1-\cos t}{t}\right)dt\right\} \tag{7.5 a}$$

$$= \tfrac{2}{9}k^4\delta^2\{1-\tfrac{1}{15}k^2+O(k^4)\} \quad (k\to 0) \tag{7.5 b}$$

$$\sim 2k^2\delta^2\{1+(1-2\gamma-2\log 2k)k^{-2}+O(k^{-4})\} \quad (k\to\infty, k\delta \ll 1), \tag{7.5 c}$$

where γ is Euler's constant. The error factor for each of (7.5*a*, *b*, *c*) is $1+O(\delta^2\log\delta)$, but not uniformly as $k\to\infty$. The approximation (7.5*b*) is within 1% (5%) of (7.5*a*) for $k < 1{\cdot}4\,(2{\cdot}0)$. The approximation (7.5*c*) is within 1% of (7.5*a*) for $k > 2{\cdot}6$. The numerical values of $C_D/k^2\delta^2$ given by (7.5*a*) are compared with the corresponding values for a sphere in figure 6.

The drag implied by (7.5*c*) in the low-speed limit is

$$D \sim (4/\pi)\rho\Omega^2 S_0^2 = (9/4\pi)(\rho\Omega^2V^2/l^2) \quad (\delta\to 0,\ k\to\infty), \tag{7.6}$$

where S_0 is the frontal area. It is $12\tfrac{1}{2}$% larger (19% smaller) than that given by (5.7), for the body described by (5.6), for equal volume (frontal area). It is 63% larger than that given by (5.9), for the body prescribed by (5.8), for equal frontal area. We infer from (6.21) that the error factor for (7.6) is $1+O(\kappa^2\log\kappa)$. This factor is likely to differ substantially from unity in the upper portion of the range $0<\kappa<\kappa_c$ [cf. the drag on a slender, semi-elliptical obstacle in a stratified shear flow (Miles & Huppert 1969)].

We turn now to the limit $k\to\infty$ and the determination of κ_c. Substituting (7.1) into (6.9), we obtain

$$\omega'(x) = -\tfrac{1}{2}\pi\kappa^2 x/\Omega^2(x). \tag{7.7}$$

Expressing $\omega_*(x)$ in terms of $\omega(x)$ by integration by parts, and invoking the fact that Ω is an even function of x, we obtain

$$\omega_*(x) = -\tfrac{1}{2}\kappa^2\int_0^1 \frac{\xi}{\Omega^2(\xi)}\log\left|\frac{x+\xi}{x-\xi}\right|d\xi, \tag{7.8}$$

which is an odd function of x that is negative-definite in $x=(0,1)$ and vanishes at $|x|=0$ and ∞. Substituting (7.1) and (7.8) into (6.16) and (6.19), we obtain

$$\mu(x) = \frac{2e^{-\omega_*(x)}}{\pi}\int_0^1\{x\cosh\omega_*(\xi)+\xi\sinh\omega_*(\xi)\}\frac{1-\xi^2}{x^2-\xi^2}\frac{d\xi}{\Omega(\xi)} \tag{7.9}$$

and

$$\mathscr{A}_1 = \tfrac{1}{2}\kappa^2\int_0^1\{(1-\xi^2)/\Omega(\xi)\}\cosh\omega_*(\xi)\,d\xi. \tag{7.10}$$

Substituting (7.9) into (6.20), we obtain

$$\kappa^2 e^{-\omega_*(1)} \int_0^1 \{\cosh \omega_*(\xi) + \xi \sinh \omega_*(\xi)\} \frac{d\xi}{\Omega(\xi)} = 4\cdot 95, \tag{7.11}$$

the numerical solution of which yields $\kappa_c = 1\cdot 94$.

We approximate $\mathscr{A}_1$ with the aid of the empirical approximations

$$\Omega(\kappa\eta) \doteqdot (1 - \tfrac{1}{4}\kappa\eta)^{-1}, \tag{7.12}$$

which is within 5 % (10 %) of the exact value for $\kappa\eta \leqslant 1\cdot 8\,(2\cdot 0)$, and

$$\omega_*(x) \doteqdot -bx \quad (0 \leqslant x \leqslant 1), \tag{7.13}$$

where we may choose b in such a way as to bound $-\omega_*$, and hence $\mathscr{A}_1$, from either above or below. Substituting (7.12) and (7.13) into (7.10), we obtain

$$\mathscr{A}_1 \doteqdot (\kappa/b)^2 \{\cosh b - b^{-1} \sinh b - (3\pi\kappa/16)\, I_2(b)\} \tag{7.14a}$$

$$\doteqdot \tfrac{1}{3}\kappa^2(1 + \tfrac{1}{10}b^2 + \ldots) - \frac{3\pi}{128}\kappa^3(1 + \tfrac{1}{12}b^2 + \ldots) \quad (b \to 0), \tag{7.14b}$$

where I_2 is a modified Bessel function. Substituting (7.12) into (7.8), we obtain

$$-\omega_*(x) \doteqdot \tfrac{1}{4}\kappa^2 \bigg\{ (2 - \tfrac{1}{2}\pi\kappa + \tfrac{5}{48}\kappa^2)\,x + (\tfrac{1}{3}\pi\kappa - \tfrac{1}{16}\kappa^2)\,x^3 + (1 - x^2)\,[1 + \tfrac{1}{32}\kappa^2(1 - x^2)] \log\left(\frac{1+x}{1-x}\right) \bigg\}, \tag{7.15}$$

which is bounded by $b_{\pm}\, x$, where

$$b_+ = \kappa^2(1 - \tfrac{1}{8}\pi\kappa + \tfrac{1}{24}\kappa^2), \quad b_- = \tfrac{1}{2}\kappa^2(1 - \tfrac{1}{12}\pi\kappa + \tfrac{1}{48}\kappa^2) \quad (0 \leqslant \kappa \leqslant 2). \tag{7.16a, b}$$

Substituting these bounds into (7.14a), we obtain results that differ by less than 1 % (10 %) for $\kappa = 1(2)$.† The mean value of these two results, which is within roughly 5 % of the exact result for $\kappa \leqslant 2$, is compared with the corresponding result for a sphere in figure 4. The ratio of $\mathscr{A}_1$ for a sphere to $\mathscr{A}_1$ for a slender ellipsoid of the same frontal area varies from $1\cdot 5$ at $\kappa = 0$ to $1\cdot 2$ at $\kappa = 2$; the upstream influence of the slender ellipsoid at a given, axial distance forward of the stagnation point is much larger than that of a sphere of the same frontal area in consequence of the fact that the dimensionless distance x in (2.19) is referred to the half-length of the body.

No experimental measurements appear to be available for ellipsoids in rotating flows; however, the approximately ellipsoidal shape of the *forward* wake of stagnant fluid in Maxworthy's (1969) experiments with spheres invites a comparison with the theoretical predictions of the upstream influence for equivalent ellipsoids.‡ Let L be the length of the forward wake and a the radius of the sphere in Maxworthy's experiments; then, by hypothesis,

$$l = L + a, \quad \delta = a/(L + a). \tag{7.17a, b}$$

† It is evident from (7.14b) that $\mathscr{A}_1$ is relatively insensitive to small changes in b within the range of interest. This is not true for the integral of (7.11), for which the approximation (7.13) is inadequate.

‡ This comparison tacitly assumes that the downstream wake has only a secondary influence on the upstream flow. It is clearly of dominant importance for the drag, for which we attempt no such comparison. It appears likely that the drag measured by Maxworthy is associated primarily with viscous separation, rather than lee waves; the relative importance of these two effects may be quite different for slender bodies.

In table 1, we compare the values of $\mathscr{A}_1$ determined from (*a*) figure 4 by interpolation with respect to δ with (*b*) those determined by plotting Maxworthy's observed values of u_1 *versus* the ratio of the upstream distance to $L+a$ ($-x$ in the present notation) on logarithmic paper and comparing straight-line fits to these data with (2.18). We make this comparison only for $\kappa = 1{\cdot}74$ and $\kappa = 2{\cdot}16$, since the remaining values of κ for which Maxworthy reports results are all much larger than κ_c. The agreement is within the experimental scatter for both the interpolated and extreme ($\delta = 0$ or 1) values of $\mathscr{A}_1$.

κ	L/a	δ	$\mathscr{A}_1$ (theory)	$\mathscr{A}_1$ (experiment)
1·74	1·2	0·45	$0{\cdot}76 \pm 0{\cdot}04$	$0{\cdot}8 \pm 0{\cdot}1$
2·16	1·6	0·38	$1{\cdot}14 {+0{\cdot}21 \atop -0{\cdot}13}$	$1{\cdot}2 \pm 0{\cdot}3$

TABLE 1. Comparison between the theoretical and experimental values of the upstream-influence coefficient for two ellipsoids. The upper and lower bounds on the theoretical values of $\mathscr{A}_1$ correspond to the curves labelled $\delta = 1$ and $\delta = 0$, respectively, in figure 4.

Maxworthy also reports radial profiles of u_1 that exhibit the oscillatory behaviour predicted by (2.16). These profiles gradually spread with increasing $|x|$, presumably in consequence of viscous effects, but it appears reasonable to make a comparison with the inviscid prediction at $|x| = 5a$. The observed values of kr at which $u_1 = 0$ are 2·6 and 2·5 for $k = 1{\cdot}74$ and 2·16, respectively; the theoretical value is $kr = 2{\cdot}4$. The corresponding values of kr at which u_1 exhibits its first minimum are 4·2 and 4·5, which compare with the theoretical value of 3·8.

The comparisons of the last two paragraphs suggest that an inviscid model may be adequate for the prediction of the upstream influence of a slender body in a rotating flow for $\kappa < 2$ and $\delta < 0{\cdot}5$.

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REFERENCES

BATCHELOR, G. K. 1956 On steady laminar flow with closed streamlines at large Reynolds number. *J. Fluid Mech.* **1**, 177–90.

COURANT, R. & HILBERT, D. 1962 *Methods of Mathematical Physics*, Vol. 2. New York: Interscience.

FRAENKEL, L. E. 1956 On the flow of rotating fluid past bodies in a pipe. *Proc. Roy. Soc.* A **233**, 506–26.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids.* Cambridge University Press.

HANDELSMAN, R. A. & KELLER, J. B. 1967 Axially symmetric potential flow around a slender body. *J. Fluid Mech.* **28**, 131–47.

HUPPERT, H. E. & MILES, J. W. 1969 Lee waves in a stratified flow. Part 3. Semi-elliptical obstacle. *J. Fluid Mech.* **35**, 481–96.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LIGHTHILL, M. J. 1967 On waves generated in dispersive systems by travelling forcing effects, with applications to the dynamics of rotating fluids. *J. Fluid Mech.* **27**, 725–52.

LONG, R. R. 1953 Steady motion around a symmetrical obstacle moving along the axis of a rotating fluid. *J. Meteor.* **10**, 197–203.

LONG, R. R. 1955 Some aspects of the flow of stratified fluids. III. Continuous density gradients. *Tellus*, **7**, 341–57.

LONG, R. R. 1962 Velocity concentrations in stratified fluids. *J. Hydraulics Div., Proc. A.S.C.E.* **88**, 9–26.

MAXWORTHY, T. 1969 The flow created by a sphere moving along the axis of a rotating, slightly viscous fluid: a Proudman–Taylor problem. *J. Fluid Mech.* (sub judice).

MILES, J. W. 1968 Lee waves in a stratified flow. Part. 2. Semicircular obstacle. *J. Fluid Mech.* **33**, 803–14.

MILES, J. W. & HUPPERT, H. E. 1969 Lee waves in a stratified flow. Part 4. Perturbation approximations. *J. Fluid Mech.* **35**, 497–526.

MORAN, J. P. 1963 Line source distributions and slender-body theory. *J. Fluid Mech.* **17**, 285–304.

MUNK, MAX M. 1934 Aerodynamics of airships. Article in Vol. 6 of *Aerodynamic Theory* (W. F. Durand, Ed.). Berlin: Springer.

MUSKHELISHVILI, N. I. 1953 *Singular Integral Equations*. Groningen: Noordhoff.

PÖLYA, G. & SZEGÖ, G. 1951 *Isoperimetric Inequalities in Mathematical Physics*. Princeton University Press.

RAYLEIGH, LORD 1871 On the light from the sky, its polarization and colour. *Phil. Mag.* **41**, 107–20; *Scientific Papers*, vol. 1, pp. 87–103.

RAYLEIGH, LORD 1897 On the incidence of aerial and electric waves upon small obstacles. *Phil. Mag.* **44**, 28–52; *Scientific Papers*, vol. 4, pp. 303–26.

RAYLEIGH, LORD 1916 On the dynamics of revolving fluids. *Proc. Roy. Soc.* A **93**, 148–54; *Scientific Papers*, **6**, 447–53.

SQUIRE, H. B. 1956 Rotating fluids. Article in *Surveys in Mechanics* (Edited by G. K. Batchelor & R. M. Davies). Cambridge University Press.

STEWARTSON, K. 1958 On the motion of a sphere along the axis of a rotating fluid. *Quart. J. Mech. Appl. Math.* **11**, 39–51.

STEWARTSON, K. 1968*a* On inviscid flow of a rotating fluid past an axially-symmetric body using Oséen's equations. *Quart. J. Mech. Appl. Math.* **21**, 353–73.

STEWARTSON, K. 1968*b* (personal communication).

TAYLOR, G. I. 1928 The energy of a body moving in an infinite fluid, with an application to airships. *Proc. Roy. Soc.* A **120**, 13–21.

TITCHMARSH, E. C. 1948 *Introduction to the Theory of Fourier Integrals*. Oxford University Press.

TRUSTRUM, K. 1964 Rotating and stratified fluid flow. *J. Fluid Mech.* **19**, 415–32.

VON KÁRMÁN, Th. 1927 Berechnung der Druckverteilung an Luftschiffkörpern. *Abhandlungen aus dem Aerodynamischen Institut an der Technischen Hochschule, Aachen*, **6**, 3–17.

VON KÁRMÁN, TH. 1936 The problem of resistance in compressible fluids. *Reale Accademia d'Italia*, **14**, 5–59.